FST 10.3 Notes
Topic: The Binomial Theorem
GOAL
Explain the connections between combinations, Pascal's Triangle, and binomial coefficients and use the binomial theorem to solve certain counting problems in preparation for the next lesson.

SPUR Objectives

A Expand binomials using the Binomial Theorem.

D Interpret and describe properties of binomial coefficients combinatorially and algebraically.

K Represent combinations and binomial coefficients by Pascal's Triangle.

Vocabulary

expansion of $(x+y)^n$
binomial coefficients

binomial coefficients The coefficients in the expansion of $(x + y)^n$; the combinations ${}_nC_k$.

Mental Math

Multiply.

a. $(4 + 8)(4 + 8)$ $(12)(12) = 144$

b. $(x + y)(x + y)$ $x^2 + 2xy + y^2$

c. $(40 + 1)(40 + 1)$ $(41)(41) = 1681$

	x	y
x	x^2	xy
y	xy	y^2

Review Pascal Triangle

Row 0 1 → ${}_0C_0$

Row 1 1 1 → ${}_1C_0$ ${}_1C_1$

Row 2 1 2 1 → ${}_2C_0$ ${}_2C_1$ ${}_2C_2$

Row 3 1 3 3 1 → ${}_3C_0$ ${}_3C_1$ ${}_3C_2$ ${}_3C_3$

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

a) How do we get to the next row?

Add 2 numbers above

b) How are the values in the triangle related to ${}_nC_r$?

n = row #

r = begins with 0, counts up by 1 and ends when it matches the row #

Review Expanding of binomials

1) $(x+y)^0 = 1$

2) $(x+y)^1 = x+y$

3) $(x+y)^2 = (x+y)(x+y) = x^2+2xy+y^2$

4) $(x+y)^3 = \underbrace{(x+y)(x+y)}(x+y)$

	x^2	$2xy$	y^2
x	x^3	$2x^2y$	$1xy^2$
y	$1x^2y$	$2xy^2$	y^3

$(x^2+2xy+y^2)(x+y) = 1x^3+3x^2y+3xy^2+1y^3$

If we try to expand $(x+y)^4$ it gets more cumbersome and difficult algebraically. The binomial theorem allows us to expand a power of a binomial.

$(x+y)^3 = {}_3C_0x^3y^0 + {}_3C_1x^2y^1 + {}_3C_2x^1y^2 + {}_3C_3x^0y^3$

Binomial Theorem

For any nonnegative integer n,

$$(x+y)^n = {}_nC_0x^ny^0 + {}_nC_1x^{n-1}y^1 + {}_nC_2x^{n-2}y^2 + \cdots + {}_nC_kx^{n-k}y^k + \cdots + {}_nC_nx^0y^n = \sum_{k=0}^{n} {}_nC_kx^{n-k}y^k.$$

****Each row of Pascal's triangle gives the coefficients for the expansion of $(x+y)^n$ for positive integers n.**

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

For example, $(x+y)^4 = {}_4C_0x^4y^0 + {}_4C_1x^3y^1 + {}_4C_2x^2y^2 + {}_4C_3x^1y^3 + {}_4C_4y^4x^0$
$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$

Using the Binomial Theorem expand the following expressions.

Expand $(x+y)^5 = {}_5C_0x^5y^0 + {}_5C_1x^4y^1 + {}_5C_2x^3y^2 + {}_5C_3x^2y^3 + {}_5C_4x^1y^4 + {}_5C_5x^0y^5$

$1x^5y^0 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5x^1y^4 + 1x^0y^5$

Example 1

Find the power of y and the coefficient of the x^5 term in $(x+y)^9$.

power of x	power of y	Product Type	Coefficients
5	4	x^5y^4	$_9C_4$ or $_9C_5$ 126

$126x^5y^4$

Example 2

Expand $(3-4y)^5$ and check by letting y = 1.

$$_5C_0(3)^5(-4y)^0 + {}_5C_1(3)^4(-4y)^1 + {}_5C_2(3)^3(-4y)^2 + {}_5C_3(3)^2(-4y)^3 + {}_5C_4(3)^1(-4y)^4 + {}_5C_5(3)^0(-4y)^5$$

$$= 243y^0 - 1620y^1 + 4320y^2 - 5760y^3 + 3840y^4 - 1024y^5$$

Example 3

a) A coin is flipped five times. How many of the possible arrangements of heads and tails have at least two heads? 2 or more

Number of Heads	Number of Tails	Sequence Type	Arrangements	
2	3	HH TTT	$_5C_3=10$	$_5C_2$ 10
3	2	HHH TT	$_5C_2=10$	$_5C_3=10$
4	1	HHHH T	$_5C_1=5$	$_5C_4=5$
5	0	HHHHH	$_5C_0=1$	$_5C_5=1$
		Total	26	26

26 ways

Example 3

b) Find all terms in $(H+T)^5$ in which the power of H is at least two. 2 or more

Power of H	Power of T	Product Type	Coefficients
2	3	H^2T^3	$5C_3 = 10$
3	2	H^3T^2	$5C_2 = 10$
4	1	H^4T^1	$5C_1 = 5$
5	0	H^5T^0	$5C_0 = 1$

$10H^2T^3,\ 10H^3T^2,\ 5H^4T^1,\ 1H^5$